x, *of the equations* (14), *are precisely the only possible solutions of the given problem of the Calculus of Variations*; *or, the only possible solutions of the given problem of the Calculus of Variations are given by the characteristic torsion strips of the equation* (2), *regarded as a differential equation.*

We might now easily go on to set up, as we have done in Part I, § 6 for equations of the first order, the equations of the common characteristics of two partial differential equations of the second order, and the condition that they be in involution, regarding the common characteristics as the common possible solutions of two problems in the Calculus of Variations, which reduce to a single problem of the type given above with one additional auxiliary condition of the type (2). Finally, we might seek the characteristics for two differential equations of the first order in two dependent variables, and so on. These problems will, however, offer no essential difficulty to the reader, and we will not enter into a discussion of them here; the main point of the existence of a connection between the theory of characteristics and the Calculus of Variations already having been demonstrated.

SHEFFIELD SCIENTIFIC SCHOOL,
YALE UNIVERSITY.
FEBRUARY, 1903.

ERRATUM.

Page 139, line 7: instead of $q = \psi'(y)$, read $q = \phi'(y)$.

ON THE UNIFORMITY OF THE CONVERGENCE OF CERTAIN ABSOLUTELY CONVERGENT SERIES

BY MAXIME BÔCHER

IF the series

$$u_1(x) + u_2(x) + \cdots \tag{1}$$

is absolutely and uniformly convergent for the values of x in a certain interval, and if we rearrange the terms, will the resulting series necessarily be uniformly convergent? This question must be answered in the negative as the following example shows:

$$x^2 - x^2 + \frac{x^2}{1+x^2} - \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} - \frac{x^2}{(1+x^2)^2} + \cdots. \tag{2}$$

Here $\qquad S_{2n} = 0, \qquad S_{2n+1} = \dfrac{x^2}{(1+x^2)^n}.$

It is easily seen that S_{2n+1}, which reaches its maximum when $x = \pm 1/\sqrt{n-1}$, is always less than $1/n$. Accordingly (2) converges uniformly to the value zero. It is also readily shown to be absolutely convergent.

Now form the series

$$x^2 - x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} - \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^3} + \frac{x^2}{(1+x^2)^4} - \frac{x^2}{(1+x^2)^2} + \cdot\ \cdot$$

from (2) by rearranging the terms. This series, of course, also converges to the value zero, but it converges non-uniformly in any interval including the point $x = 0$. For

$$S_{3n-1} = \frac{x^2}{(1+x^2)^n}\left[1 + \frac{1}{1+x^2} + \cdot\ \cdot\ \cdot + \frac{1}{(1+x^2)^{n-2}}\right] = \frac{(1+x^2)^{n-1} - 1}{(1+x^2)^{2n-2}},$$

and when x has either of the real values

$$\pm\sqrt{\sqrt[n-1]{2} - 1}$$

we have $S_{3n-1} = 1/4$.

It is, however, easy to prove the following theorem:

If the series

$$|\, u_1(x) \,| + |\, u_2(x) \,| + \cdot\ \cdot\ \cdot$$

is uniformly convergent throughout the interval in question, (1) *will be absolutely and uniformly convergent, and will remain so no matter how the order of the terms is changed.*

Any series which can be proved uniformly convergent by Weierstrass's test (*cf.* Weierstrass, *Werke*, Vol. II, p. 202) will therefore remain uniformly convergent when its terms are rearranged.

The facts here referred to would seem to be of especial interest in relation to the subject of absolutely convergent multiple series.

In conclusion I will mention the following theorem which I had occasion to prove and to use in a slightly-different form a few years ago (*cf. Bull. Amer. Math. Soc.* May, 1898, p. 368):

If throughout the interval $a \leqq x \leqq b$ the functions $u_i(x)$ are continuous and nowhere negative, and if the function represented by the series (1) *is continuous throughout the interval $a \leqq x \leqq b$, then* (1) *converges uniformly throughout this interval.*

HARVARD UNIVERSITY, CAMBRIDGE, MASS.
JANUARY, 1903.